Matlab based observations on the solutions of Discrete Infinite System *

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Abstract

In the present paper we introduce a Matlab algorithm in order to study the behaviour of the following discrete infinite Ordinary Differential Equation (O.D.E) system:
$\dot{u_n}(t) - \delta[g(u_{n-1}(t) - 2g(u_n(t)) + g(u_{n+1}(t))] + f(u_n(t)) = 0$,
where $u_n(0) = u_{n,0}$. Here we have that $\delta > 0$ constant, $f : \mathcal{R} \to \mathcal{R}$ is a continuous function, $n \in \mathcal{Z}$ and g is a function such that $g(s) = s^m,\ m \geq 1$.

The above system result on particular cases of interest, like Porus Medium Equation and Fisher equation. Also using matlab, we give several observations on the solutions of our discrete system for a number of given fuctions.

In many cases, the results of the mathematical models of many physical phenomenons, product a special kind of differential equations like: Lattice Ordinary Differential Equations (L.O.D.E). We

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have a lot of applications for these kind of mathematical models: In biology (in calcium bursts in living shells), in propagation of action potentials in through the tissue of the cardiac shells, in DNA's transmission, in Physics (superconductivity theory and non-linear optics, Josephsson's patterns), in the mechanic's of the materials (solidity and long range attribute of the materials) and finally in technology (scheduling electronic loop) [7], [13], [16], [17] and [19].

1 Introduction and Prelimary Results

The aim of this work is to investigate the behaviour of the following initial value problem:

$$\dot{u_n}(t) - \delta[g(u_{n-1}(t) - 2g(u_n(t)) + g(u_{n+1}(t))] + f(u_n(t)) = 0, \quad (1.1)$$

$$u_n(0) = u_{n,0}. \quad (1.2)$$

In order to succed that we introduce a Matlab algorithm that evaluates terms of the sequence $u_n(t)$. We consider system (1.1)- (1.2) as an infinite dimensional system of O.D.E's in the Banach space l^2.

The above paper is the basis for many analytical and arithmetical results, with ultimate example: the local existence in time and space, periodic or quasiperiodic solutions for an expansive league (L.O.E.D), using anti-continuum limit (the validation of the case where the distance between the nodes of the lattice $h \to \infty$). We refer to [5], [14]. In order to study queries like existence and stability of solutions for L.O.D.E, we use a lot of technical varieties like: degenerate perturbing theory, asymptotic methods, non-linear theory of stability, bifurcation theory (see [13]).

During the 90's L.O.D.E where studied by the use of infinite dynamical systems. Especially initial problems with boundary conditions for partial differential equations (P.D.E) where used in order to solve this kind of equations. We typically refer that in several cases the interaction between the nodes of the lattice has a limited range and this interaction has been reconstructed by the finite difference discreteness of a Laplace differential operator. For example we refer to the following system:

$$\dot{u_{i,j}} = a[(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + (u_{i,j+1} - 2u_{i,j} + u_{i,j-1})] = f(u_{i,j}), \quad (1.3)$$

where $i, j \in \mathcal{Z}$, which turns out from very important applications (image processing, neuronic networks, dynamics of the populations).

Typical examples about existence and stability of traveling waves for L.O.D.E at the type of (1.3), we meet at the works [6], [8] [22] and [23]. Furthermore, for the existence of solitary waves for discrete equations we refer to [12], [18]. Especially for L.O.D.E we have the following system:

$$\ddot{\phi}_i = V'(\phi_{i+1} - \phi_i) - V'(\phi_i - \phi_{i-1}), \quad i \in \mathcal{Z},$$

which has been studied by E. Fermi, J. Pasta and S. Ulam. They used the previous system in their classical theory about the dynamics of Hamiltonian systems [9]. Questions about existence of global attractors for L.O.D.E where firstly answered by V. S. Afraimovich, S. N. Chow and J. K. Hale, in their paper [1]. They worked on a Duffing's type $N \times N$ network of interact oscillations. They also studied the above problem in a form equivalent to the system (1.3), using Dirichlet boundary conditions (L.O.D.E system on $\mathcal{R}^{2N}$). The existence of global attractors for the infinite L.O.D.E systems (1.3), has also been studied in the following papers: [3], [20], [21].

At the end, in this work, we will also give results about existence and uniqueness of solutions for the following systems:

$$\dot{u}_i = a[\theta(u_{i+1}) - 2\theta(u_i) + \theta(u_{i-1})] = f(u_i), \ \ i \in \mathcal{Z}, \tag{1.4}$$

or for the second order L.O.D.E:

$$\ddot{u}_i = a(u_{i+1} - u_{i-1})[(u - i + 1) - (u_i) + (u_{i-1})] + \delta L(\dot{u})_i = f(u_i), \tag{1.5}$$

where $L(\cdot)_{i \in \mathcal{Z}}$ is a discrete operator. Problems (1.4) and (1.5) (when $a, \ \theta : \mathcal{R} \to \mathcal{R}$ are non-monotonous), encounter in the mechanics of the materials (for example in dislocation of crystals). See [2], [4], [10], and [11].

The problem is well defined as it is expressed in the following theorem

Theorem 1.1 *Let $m \geq 1$ and $f : \mathcal{R} \to \mathcal{R}$ be continuous. There there exists a local in time solution for the problem (1.1)-(1.2) if $u_{n,0} \in l^2$, (see [15]).* $\diamondsuit$

Remark 1.2 *The same conclusion holds for the operator $f(u)$. We have the following typical examples: (**I**) $f(u) = u^p$, $p \geq 1$, (typical power-law nonlinearity) and (**II**) $f(u) = u(1-u)$, (Fisher equation).*$\diamondsuit$

Remark 1.3 *Theorem 2.1 holds for the particular case of interest*

$$f(s) = s^p, \ \ or \ \ f(s) = s(1-s) \ \ (Fisher \ \ equation).$$

Especially, for the case $f(s) = s^p,\ p \geq 1,$ *the local existence result holds without any restriction on the exponents* $p,\ m,$ *which is a difference concerning the continuous model (Porus Medium Equation). Porus medium equation (shortly PME), is the heat equation* $\theta_t u = \Delta(u^m),\ m > 1,$ *posed in the d-dimensional Euclidean space, with interest in the cases* $d = 1, 2, 3$. *We have that* $\Delta = \Delta_x$ *represents the Laplace operator acting on the space variables. The complete form is*

$$u_t = \Delta(|u|^{m-1}u) + f.$$

There are a number of physical applications where the simple (for $m = 1$*) PME model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Other applications have been proposed in mathematical biology, lubrication, boundary layer theory and many other fields.* $\diamondsuit$

2 Matlab based observations

The examples of this paragraph were executed on Matlab. The algorithm is based on the Initial Value Problem defined in (1.1) and (1.2) for $\delta > 1$. The number of iterations remained small due to the increasing complexity of the algorithm.

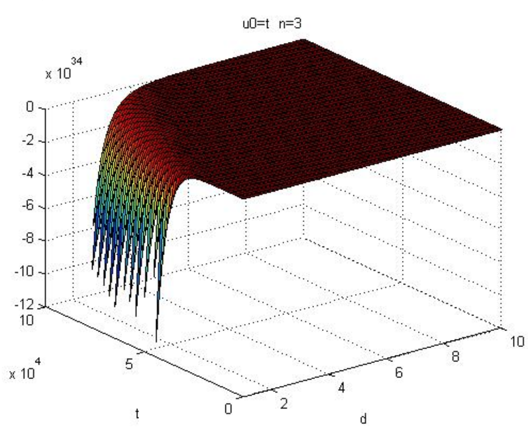

figure 1: $u_0 = t$, 3 iterations

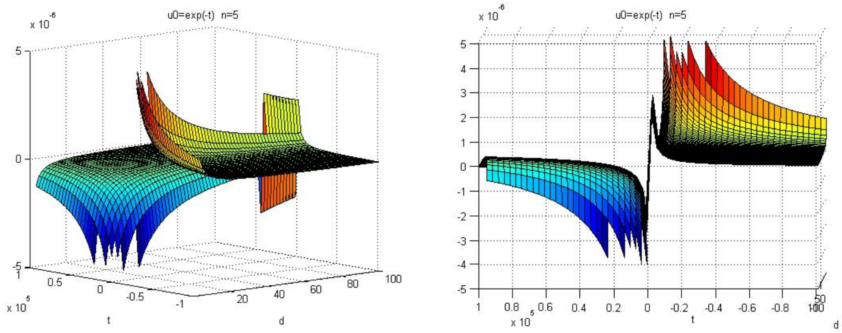

figure 2: $u_0 = e^{-t}$, 5 iterations

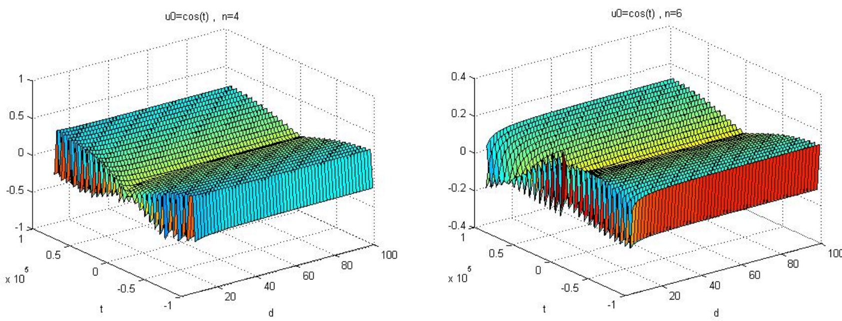

figure 3: $u_0 = \cos t$, 4 and 6 iterations

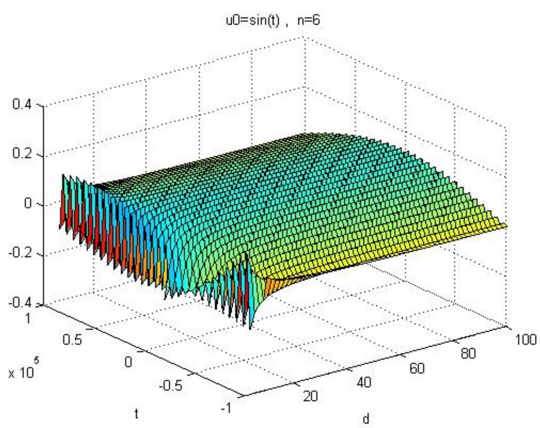

figure 4: $u_0 = \sin t$, 6 iterations

In the above examples we observe that the values are very small. The functions seem smooth as δ increases. The functions sink or explode as δ

approaches to 1. The functions seem to preserve the symmetric or skew-symmetric behavior of the initial function u_0. They also follow the wave patterns of the initial periodic functions. Successive terms seem to have similar behavior. Thus we have an indicator for the behavior of system. These observations are open issues to further investigation and theoretical confirmation.

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